Exit-problem of particles interacting with their empirical law

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Abstract

The current article is devoted to the study of a mean-field system of particles. More precisely, we solve the exit-problem of the first particle (and from any particle) from a domain on $\mathbb{R}^d$. We establish a Kramers' type law with an exit-cost which converges to a given quantity as the number of particles is large. For doing so, we slightly modify the classical assumptions on the Freidlin-Wentzell theory and we use the recent result about the large deviations from Herrmann and us. The main improvement of the paper is that it is applied without assuming any global convexity of the external force nor on the interacting potential.

Key words and phrases: Exit-problem ; Large deviations ; Interacting particle systems ; Mean-field systems

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1 Introduction

The paper is devoted to the resolution of the exit-problem of some mean-field interacting particles system. Let us briefly present the model. We consider a sequence $(X_0^i)_{i\geq 1}$ of independent and identically distributed random variables with a common law μ_0 on $\mathbb{R}^d$. Also, for any $i \in \mathbb{N}^*$, $\{B_t^i \ : \ t \in \mathbb{R}_+\}$ is a Brownian motion on $\mathbb{R}^d$ which is independent from the sequence $(X_0^i)_i$. The Brownian motions are assumed to be independent.

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Each of the particles evolves in a non-convex landscape V, that we denote as the confining potential. Moreover, each of the particles interacts with any other particle. We assume that the interaction does only depend on the distance between the two particles. We do not assume this interacting force either to be an attraction nor to be a repulsion.

In fine, the system of equations that we are interested in is the following:

$$X_t^{i,N} = X_0^i + \sigma B_t^i - \int_0^t \nabla V\left(X_s^{i,N}\right) ds - \int_0^t \frac{1}{N}\sum_{j=1}^N \nabla F\left(X_s^{i,N} - X_s^{j,N}\right) ds\,, \quad \text{(I)}$$

N being an integer that we will assume to be large in the following and σ being an arbitrarily small positive constant.

We can see the N particles in $\mathbb{R}^d$ as one "big" particle in $\mathbb{R}^{dN}$. Indeed, let us write $\mathcal{X}_t^N := \left(X_t^{1,N}, \cdots, X_t^{N,N}\right)$ and $\mathcal{B}_t^N := \left(B_t^1, \cdots, B_t^N\right)$. The process $\mathcal{B}^N$ thus is a dN-dimensional Wiener process. Equation (I) can be rewritten like so:

$$\mathcal{X}_t^N = \mathcal{X}_0^N + \sigma \mathcal{B}_t^N - N\int_0^t \nabla \Upsilon^N\left(\mathcal{X}_s^N\right) ds\,. \quad \text{(II)}$$

Here, the potential on $\mathbb{R}^{dN}$ is defined by $\Upsilon^N\left(X_1, \cdots, X_N\right) := \frac{1}{N}\sum_{i=1}^N V\left(X_i\right) + \frac{1}{2N^2}\sum_{i=1}^N\sum_{j=1}^N F\left(X_i - X_j\right)$ for any $\left(X_1, \cdots, X_N\right) \in \left(\mathbb{R}^d\right)^N$.

Consequently, the whole system of particles, $\{\mathcal{X}_t^N \ : \ t \in \mathbb{R}_+\}$, is just an homogeneous and reversible diffusion in $\mathbb{R}^{dN}$ since it evolves only trough the gradient of the potential $N\Upsilon^N$.

One may wonder why we have defined Υ^N like so, without taking into account the N. In fact, the potential Υ^N has sense when N goes to infinity. Indeed, for any sequence $\left(X_k\right)_k$ of independent and identically distributed random variables with common law μ, as N goes to infinity, the quantity $\Upsilon^N\left(X_1, \cdots, X_N\right)$ converges almost surely toward

$$\Upsilon^\infty(\mu) := \int_{\mathbb{R}^d} V(x)\mu(dx) + \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} F(x-y)\mu(dx)\mu(dy)\,.$$

The above quantity corresponds to the energy of the measure μ. It appears naturally as we study the hydrodynamical limit of the interacting particles system.

Let us look heuristically to the system of particles. One may remark that the influence of the particle number j on the particle number i is divided by N. Moreover, the two particles have independent initial random variables. Consequently, it is intuitive to hope that the trajectories of the two particles are more and more independent as N grows. Since the particles have "independent" and exchangeable trajectories, the empirical measure of the system of particles at time t, that is $\eta_t^N := \frac{1}{N}\sum_{j=1}^N \delta_{X_t^{j,N}}$, would converge to the law of the first particle, $\mathcal{L}\left(X_t^{1,N}\right)$.

However, the equation which drives the first particle in Equation (I) may be rewritten like so:

$$X_t^{1,N} = X_0^1 + \sigma B_t^1 - \int_0^t \nabla V\left(X_s^{1,N}\right) ds - \int_0^t \nabla F * \eta_t^N\left(X_s^{1,N}\right) ds\,. \quad \text{(III)}$$

Thus, to understand the behavior of the diffusion $X^{1,N}$ when N is large, it is intuitive to look at the limit diffusion that is to say

$$\begin{cases} X_t^{1,\infty} = X_0^1 + \sigma B_t^1 - \int_0^t \nabla V\left(X_s^{1,\infty}\right) ds - \int_0^t \nabla F * \mu_t^\infty \left(X_s^{1,\infty}\right) ds \\ \mu_t^\infty = \mathcal{L}\left(X_t^{1,\infty}\right) \end{cases} . \tag{IV}$$

The independence between the particles is a phenomenon denoted as "propagation of chaos". It has been studied in [Szn91, Mél96]. This also is equivalent to a coupling result which statement is the following:

$$\lim_{N\to+\infty} \mathbb{E}\left\{ \sup_{t\in[0;T]} \left|\left| X_t^{1,N} - X_t^{1,\infty} \right|\right|^2 \right\} = 0\,,$$

for any $T > 0$. Here, we do not have the supremum over the whole set $\mathbb{R}_+$. In [CGM08], without assuming the strict uniform convexity of the potentials V and F, the authors obtained a coupling over the whole set $\mathbb{R}_+$:

$$\lim_{N\to+\infty} \sup_{t\geq 0} \mathbb{E}\left\{ \left|\left| X_t^{1,N} - X_t^{1,\infty} \right|\right|^2 \right\} = 0\,.$$

One can use this coupling method in order to show the existence and the uniqueness of a solution to Equation (IV), like it has been made in [Mél96]. In [McK67, BRTV98, CGM08, HIP08], the authors prove the existence and uniqueness result by using a fixed point theorem. In [McK67], it has also been proven that the law of the diffusion at time $t > 0$ is absolutely continuous with respect to the Lebesgue measure. Moreover, its density, that we denote by $u(t,x)$ satisfies a nonlinear partial differential equation:

$$\frac{\partial}{\partial t} u = \nabla.\left\{ \frac{\sigma^2}{2} \nabla u + u\left(\nabla V + \nabla F * u\right) \right\}. \tag{V}$$

This equation is the so-called granular media equation. Thank to this equation, one can study the invariant probabilities of the McKean-Vlasov diffusion $X^{1,\infty}$. This study has been made in [BRTV98, HT10a, HT10b]. About the long-time behaviour, we refer the reader to [Tug13b, Tug13c, BGG13].

Let us present what we denote by exit-problem. We consider a domain $\mathcal{D} \subset \mathbb{R}^d$ and a diffusion

$$x_t^\sigma = x_0 + \sigma B_t - \int_0^t \nabla U\left(x_s^\sigma\right) ds\,.$$

and we introduce

$$S(\sigma) := \inf\left\{ t \geq 0 \ : \ x_t^\sigma \in \mathcal{D} \right\}$$

the first hitting-time of x^σ to the domain $\mathcal{D}$. Then, we define

$$\tau(\sigma) := \inf\left\{ t \geq S(\sigma) \ : \ x_t^\sigma \notin \mathcal{D} \right\}$$

the first exit-time of x^σ from the domain $\mathcal{D}$. The exit-problem in the small-noise limit consists of two questions. What is the exit-time $\tau(\sigma)$ for σ going to 0? What is the exit-location $x^\sigma_{\tau(\sigma)}$ for σ going to 0?

The subject of this article is to study these kind of questions. They have been solved by Freidlin and Wentzell for homogeneous diffusions. See [DZ98, FW98] for a complete review. The typical result is the following:

Theorem 1.1. *We assume that $\mathcal{G}$ satisfies the following properties.*

1. *The unique critical point of the potential U in the domain $\mathcal{G}$ is a_0. Furthermore, for any $y_0 \in \mathcal{G}$, for any $t \in \mathbb{R}_+$, we have $y_t \in \mathcal{G}$ and moreover $\lim_{t\to+\infty} y_t = a_0$ with*
$$y_t = y_0 - \int_0^t \nabla U(y_s)ds\,.$$
2. *For any $y_0 \in \partial\mathcal{G}$, y_t converges toward a_0.*
3. *The quantity $H := \inf_{z\in\partial\mathcal{G}} \left(U(z) - U(a_0)\right)$ is finite.*

By $\tau_{\mathcal{G}}(\sigma)$, we denote the first exit-time of the diffusion x^σ from the domain $\mathcal{G}$. Then, for any $\delta > 0$, we have
$$\lim_{\sigma\to0} \mathbb{P}\left\{e^{\frac{2}{\sigma^2}(H-\delta)} < \tau_{\mathcal{G}}(\sigma) < e^{\frac{2}{\sigma^2}(H+\delta)}\right\} = 1\,.$$
Furthermore, if $\mathcal{N} \subset \partial\mathcal{G}$ is such that $\inf_{z\in\mathcal{N}} U(z) > \inf_{z\in\partial\mathcal{G}} U(z)$, we know that the diffusion x^σ does not exit $\mathcal{G}$ by $\mathcal{N}$ with high probability:
$$\lim_{\sigma\to0} \mathbb{P}\left\{x^\sigma_{\tau_{\mathcal{G}}(\sigma)} \in \mathcal{N}\right\} = 0\,.$$

We do not provide the proof which can be found in [DZ98].

In [Tug13c], we have obtained a similar result (already obtained in [HIP08]) for the self-stabilizing diffusion (IV). To do so, we establish a Kramers' type law for the first particle of the mean-field system of particles. In this previous work, both the confining potential and the interacting potential are assumed to be convex.

In the current paper, we remove the hypothesis of global convexity, which is the main improvement.

We can not apply the method of the article [Tug13c]. Indeed, without global convexity assumptions, the domains that we are interested in do not satisfy the hypotheses of Theorem 1.1.

Consequently, in order to obtain the exit-problem of one particle (the first one or anyone) of the mean-field system of particles from a domain, we will slightly modify Theorem 1.1. The idea will be to obtain a "new" theorem which does not require any hypothesis on the domain.

The main difficulty thus is to compute the exit-cost:
$$\inf_{\partial\mathcal{D}_N} N\Upsilon^N - N\Upsilon^N(a_0,\cdots,a_0)\,,$$

$\mathcal{D}_N$ being a domain of $\mathbb{R}^{dN}$ and a_0 a wells of the confining potential V such that $(a_0, \cdots, a_0)$ is the unique wells of Υ^N on $\mathcal{D}_N$.

In [Tug13c], we have directly computed the exit-cost. It was a difficult computation and it was strongly based on the convexity of the potentials. One could obtain the exit-cost of any domain in which the potential V is convex. However, it is a strong restriction about the domains on which we can use the theorem. Consequently, we proceed in another way. We use the recent results in [HT14] about large deviations. Typically, we have proven that the exit-cost of $X^{1,\infty}$ from a domain $\mathcal{D}$ is equal to the limit of the exit-cost of $X^{1,N}$ from $\mathcal{D}$, as N goes to infinity. Thus, we just need to compute the exit-cost of $X^{1,\infty}$ from the domain $\mathcal{D}$, which is a problem in finite dimension.

We do not give the main results now but typically, we obtain a Kramers' type law for the first exit-time of the first particle (and for any particle of the system of particles) from a domain containing a unique critical point $(a, \cdots, a)$ of Υ^N, a being a wells of V. And, the exit-cost H_N satisfies $\lim_{N\to\infty} H_N = \inf_{z\in\partial\mathcal{D}} V(z) + F(z-a) - V(a)$.

We now give the assumptions of the paper. First, we give the hypotheses on the confining potential V.

Assumption (A-1): *V is a $\mathcal{C}^2$-continuous function.*

Assumption (A-2): *For all $\lambda > 0$, there exists $R_\lambda > 0$ such that $\nabla^2 V(x) > \lambda$, for any $||x|| \geq R_\lambda$.*

We can observe that under assumptions (A-1) and (A-2), there exist a convex potential V_0 and $\theta \in \mathbb{R}$ such that $V(x) = V_0(x) - \frac{\theta}{2}||x||^2$.

Assumption (A-3) *The gradient ∇V is slowly increasing: there exist $m \in \mathbb{N}^*$ and $C > 0$ such that $||\nabla V(x)|| \leq C\left(1 + ||x||^{2m-1}\right)$, for all $x \in \mathbb{R}$.*

This assumption together with the same kind of assumptions on F ensure us that there is a global solution if some moments of μ_0 are finite.

Let us present now the assumptions on the interaction potential F:

Assumption (A-4): *There exists a function G from $\mathbb{R}_+$ to $\mathbb{R}$ such that $F(x) = G(||x||)$.*

Assumption (A-5): *G is an even polynomial function such that $\deg(G) =: 2n \geq 2$ and $G(0) = 0$.*

This hypothesis is used for simplifying the study of the invariant probabilities. Indeed, see [HT10a, HT10b, HT12, Tug13a, Tug12a], the research of an invariant probability is equivalent to a fixed-point problem in infinite dimension. Nevertheless, under Assumption (A-5), it reduces to a fixed-point problem in finite dimension.

We introduce the constant $\alpha := \inf_{\mathbb{R}} G''$.

The paper is organized as follows. We finish the introduction by discussing a little with a more general setting. In a first section, we prove the existence of a global solution on $\mathbb{R}_+$ to Equation (I) and we give some results about the geometry of the potential $N\Upsilon^N$ in function of the potential V. Then, we give first results about large deviations. In particular, we establish that the diffusion exits from the domain of attraction of a wells uniformly with respect to the number of particles under suitable assumptions, linked to the small-noise limit of invariant probabilities for McKean-Vlasov diffusions. In a fourth section, we provide a result of large deviations which is linked to the Freidlin-Wentzell theory. Finally, we give the main results. In other words, we solve the exit-problem of any particle in both attractive and repulsive case and we solve the exit-problem from a tagged particle, the first one.

We restrict ourselves to a simple case to the sake of simplicity for the computations. However, the arguments may be used for more general setting. As restriction, we ask the particles to be exchangeables. The developed method could also be extended to a drift which depends in a non-linear way of the empirical measure of the system of particles.

We are interested in a system of N diffusions $X^{1,N,\sigma},\cdots,X^{N,N,\sigma}$ which satisfy the following equation:

$$\left\{\begin{array}{l} X_t^{i,N} = x_0 + \sigma B_t^i - \int_0^t \nabla V\left(X_s^{i,N}\right)ds - \int_0^t A\left(X_s^{i,N};\eta_s^N\right)ds \\ \eta_t^N := \frac{1}{N}\sum_{j=1}^N \delta_{X_t^{j,N}} \end{array}\right. . \tag{VI}$$

Here, the probability measure η_t^N is the empirical measure of the whole particles system $\left(X^{1,N},\cdots,X^{N,N}\right)$ at time t. And, A is a general function from $\mathbb{R}^d \times \mathcal{P}\left(\mathbb{R}^d\right)$ to $\mathbb{R}^d$. We assume that the function A has the following form:

$$A(x_1;\mu) = \int_{\mathbb{R}^d} A_2(x_1,x_2)\mu(dx_2)\,. \tag{VII}$$

With $A_2(x,y) := \nabla F(x-y)$, the system of Equations (VI) corresponds to a system of particles in mean-field interaction.

We also assume in this work that the dynamic in (I) derives from a potential $N\Upsilon^N$. Let us mention that some system of interacting particles do not derive from a potential. Let us give an exemple of such system:

$$Y_t^{i,N,\sigma} = x_0 + \sigma B_t^i - \int_0^t \nabla V\left(Y_s^{i,N,\sigma}\right)ds - \frac{1}{N}\sum_{j=1}^N\int_0^t \nabla V\left(Y_s^{j,N,\sigma}\right)ds - l(t)\,.$$

Here V is a potential which is convex at infinity and l is a nondecreasing function. By taking the hydrodynamical limit, we obtain the non-linear diffusion

$$dY_t = \sigma dB_t - \left(\nabla V\left(Y_t\right) - \overline{V}_t\right)dt - l'(t)dt$$

with $\overline{V}_t := \mathbb{E}\left\{\nabla V\left(Y_t\right)\right\}$. This diffusion is the probabilistic interpretation of the nonlinear partial differential equation

$$\frac{\partial}{\partial t}u^\sigma = \mathrm{div}\left\{\frac{\sigma^2}{2}\nabla u^\sigma + \left(\nabla V - \int_{\mathbb{R}}\nabla V(x)u^\sigma(t,x)dx\right)u^\sigma\right\}.$$

This equation characterizes the charge and the discharge of the cathod in lithium battery. See [DGGHJ11, DGH11]. In this setting, the dynamic does not derive from a potential.

2 Preliminaries

We begin this paper by justifying the existence of a global non-explosive solution to Equation (I). On this purpose, we recall Theorem 10.2.2 in [SV79].

Proposition 2.1. *Let k be any positive integer. Let f be a function from $\mathbb{R}_+ \times \mathbb{R}^k$ to $\mathbb{R}^k$. We assume that the function f is locally Lipschitz-continuous, uniformly with respect to the time on each compact and satisfies the inequality*

$$\sup_{0\leq t\leq T} ||f(t,0)|| < \infty \,, \tag{1}$$

for any $T > 0$. Moreover, we assume that there exists $R > 0$ such that

$$\langle \mathcal{X} ; f(t,\mathcal{X})\rangle \leq 0 \,, \tag{2}$$

for any $\mathcal{X} \in \mathbb{R}^k$ which verifies $||\mathcal{X}|| \geq R$.
Then, if W is a Brownian motion, the stochastic differential equation,

$$X_t = X_0 + \sigma W_t + \int_0^t f\left(s, X_s\right) ds \,,$$

admits a unique non-explosive strong solution for any initial random variable X_0.

We now precise the norm that we use in this work. On $\mathbb{R}^d$, we use the euclidean norm.

Definition 2.2. *Let N be a positive integer. On $\mathbb{R}^{dN}$, we use the norm $||\,.\,||_N$ defined by*

$$||\mathcal{X}||_N^2 := \frac{1}{N}\sum_{i=1}^N |X_i|^2 \,,$$

with $\mathcal{X} := (X_1, \cdots, X_N) \in \mathbb{R}^{dN}$. This norm derives from the following scalar product:

$$\langle \mathcal{X} \,;\, \mathcal{Y}\rangle_N := \frac{1}{N}\sum_{i=1}^N X_i Y_i \,,$$

for any $\mathcal{X} = (X_1, \cdots, X_N) \in \mathbb{R}^{dN}$ and $\mathcal{Y} = (Y_1, \cdots, Y_N) \in \mathbb{R}^{dN}$.

Let us observe that this norm has sense as N is large. Indeed, let $(X_i)_{i\geq 1}$ (resp. $(Y_i)_{i\geq 1}$) be a sequence of independent and identically distributed random variables with common law μ_0 (resp. ν_0). By $\mathcal{Y}_\tau$, we denote the vector $\left(Y_{\tau(1)}, \cdots, Y_{\tau(n)}\right)$ for any permutation τ. Thus, the quantity

$$\inf_{\tau\in\mathcal{S}_N} ||\mathcal{X} - \mathcal{Y}_\tau||_N$$

converges almost surely toward $\mathbb{W}\left(\mu_0 ; \nu_0\right)$, the Wasserstein distance between the two measures μ_0 and ν_0.

We now are able to provide the existence of Diffusion (I).

Theorem 2.3. *Let N be any positive integer. Under the hypotheses of the article, the stochastic differential system of Equations* (I) *admits a unique strong solution* $\left(\mathcal{X}_t^N\right)_{t\geq 0}$.

Proof. We only sketch the proof since it is classical. Let us assume that $\alpha = \inf_{\mathbb{R}} G''$ is positive. The case $\alpha < 0$ can be solved in a similar way.
We put $f(t,\mathcal{X}) := -N\nabla\Upsilon^N(\mathcal{X})$. The ith coordinate is equal to

$$f(t,\mathcal{X})_i := -\nabla V\left(X_i\right) - \frac{1}{N}\sum_{j=1}^N \nabla F\left(X_i - X_j\right)\,,$$

where $x \mapsto F(x) - \frac{\alpha}{2}x^2$ is convex.
The function f does not depend on the time. According to the hypotheses, the function f is locally Lipschitz. We also remark $f(t,(0,\cdots,0))_i = -\nabla V(0)$ which implies Inequality (1) that is the boundedness of $f(t,(0,\cdots,0))$. To apply Proposition 2.1, it is now sufficient to prove Inequality (2).
According to the assumptions, we have the majoration

$$\frac{1}{N}\sum_{i=1}^N \langle X_i\,;\,\nabla V(X_i)\rangle \geq \theta_2\left|\left|\mathcal{X}\right|\right|_N^2 - \theta_0\,,$$

θ_0 and θ_2 being positive real. This quantity is positive if $\left|\left|\mathcal{X}\right|\right|_N$ is larger than $\sqrt{\frac{\theta_0}{\theta_2}}$. Also, we can write

$$\frac{1}{N^2}\sum_{i=1}^N\sum_{j=1}^N \langle X_i\,;\,\nabla F\left(X_i - X_j\right)\rangle = \frac{1}{2N^2}\sum_{i=1}^N\sum_{j=1}^N \langle X_i - X_j\,;\,\nabla F\left(X_i - X_j\right)\rangle \geq 0\,.$$

We deduce the negativity of $\langle \mathcal{X}\,;\,f(t,\mathcal{X})\rangle_N$ if $\left|\left|\mathcal{X}\right|\right|_N$ is sufficiently large, which achieves the proof. $\square$

We now present a result about the geometry of the potential Υ^N from the one of the potential V.

Proposition 2.4. *Let a be a real such that $\nabla V(a) = 0$. Then $\overline{a} := (a,\cdots,a)$ is a critical point of the potential Υ^N. Moreover, if $\beta := G''(0)$ is positive, we have*
i) *If $\nabla^2 V(a) > 0$, $\overline{a}$ is a local minimizer of the potential Υ^N.*
ii) *If $\nabla^2 V(a) < 0$ and $\nabla^2 V(a) + \beta > 0$, $\overline{a}$ is a saddle-point of the potential Υ^N and its hessian has the signature $(N-1,1)$.*
iii) *If $\nabla^2 V(a) + \beta < 0$, $\overline{a}$ is a local maximizer of the potential Υ^N.*

And, if β is negative, we have
i) *If $\nabla^2 V(a) + \beta > 0$, $\overline{a}$ is a local minimizer of the potential Υ^N.*
ii) *If $\nabla^2 V(a) + \beta < 0$ and $\nabla^2 V(a) > 0$, $\overline{a}$ is a saddle-point of the potential Υ^N and its hessian has the signature $(1,N-1)$.*
iii) *If $\nabla^2 V(a) < 0$, $\overline{a}$ is a local maximizer of the potential Υ^N.*

Proof. We proceed only in the case where β is positive. We compute the gradient of the potential Υ^N. By definition, we have:

$$\frac{\partial}{\partial X_i}\Upsilon^N\left(X_1;\cdots;X_N\right)=\frac{1}{N}\nabla V(X_i)+\frac{1}{N^2}\sum_{j=1}^N\nabla F\left(X_i-X_j\right)\,.$$

So, if $a\in\mathbb{R}$ is such that $\nabla V(a)=0$, we have:

$$\frac{\partial}{\partial X_i}\Upsilon_+^N\left(a;\cdots;a\right)=\frac{1}{N}\nabla V(a)+\frac{1}{N^2}\sum_{j=1}^N\nabla F\left(a-a\right)=\frac{1}{N}\nabla V(a)=0\,.$$

We deduce that $\overline{a}$ is a critical point of the potential Υ^N.
We now look at the hessian:

$$\frac{\partial^2}{\partial X_i^2}\Upsilon^N(X_1;\cdots;X_N)=\frac{1}{N}\nabla^2V(X_i)+\frac{\beta}{N}\left(1-\frac{1}{N}\right)$$

and

$$\frac{\partial^2}{\partial X_i\partial X_j}\Upsilon^N\left(X_1;\cdots;X_N\right)=-\frac{\beta}{N^2}\,,$$

if $i\neq j$. Consequently, we have

$$\nabla^2\Upsilon^N\left(a;\cdots;a\right)=\frac{1}{N}\left(\;\left(c_{i,j}\right)\;\right)_{\substack{1\leq i\leq N\\1\leq j\leq N}}$$

with $c_{i,i}:=\nabla^2V(a)+\beta\left(1-\frac{1}{N}\right)$ and $c_{i,j}:=-\frac{\beta}{N}$ if $i\neq j$.
Some basic computations in linear algebra give two eigenvalues: $\lambda:=\frac{1}{N}\nabla^2V(a)$ with an eigenspace which dimension is 1 and $\mu:=\frac{1}{N}\nabla^2V(a)+\frac{\beta}{N}$ with an eigenspace which dimension is $N-1$. This achieves the proof. $\square$

3 First results

Let N be any positive integer. In the current work, we deal with the time-homogeneous diffusion $\mathcal{X}^N$,

$$\mathcal{X}_t^N=\overline{x_0}+\sigma\mathcal{B}_t^N-N\int_0^t\nabla\Upsilon^N\left(\mathcal{X}_s^N\right)ds\,.$$

with $\overline{x_0}:=(x_0,\cdots,x_0)$ and Υ^N is a potential of $\mathbb{R}^N$. We now introduce its first exit-time from any domain.

Definition 3.1. *Let $\mathcal{D}$ be any open set of $\left(\mathbb{R}^d\right)^N$. We define the first entering time of $\mathcal{X}^N$ in $\mathcal{D}$ by*

$$E_{\mathcal{D}}(\sigma,N):=\inf\left\{t\geq0\;:\;\mathcal{X}_t^N\in\mathcal{D}\right\}\,.$$

We define its first exit-time from $\mathcal{D}$ by

$$\tau_{\mathcal{D}}(\sigma,N):=\inf\left\{t\geq E_{\mathcal{D}}(\sigma,N)\;:\;\mathcal{X}_t^N\notin\mathcal{D}\right\}\,.$$

Intuitively, if σ is small, the diffusion $\mathcal{X}^N$ is close to a deterministic system that we now introduce.

Definition 3.2. *We consider the deterministic system*

$$\varphi_t\left(\overline{x_0}\right) = \overline{x_0} - N\int_0^t \nabla\Upsilon^N\left(\varphi_s\left(\overline{x_0}\right)\right)ds$$

Before giving new results about the diffusion $\mathcal{X}^N$, we present some classical large deviations results.

Proposition 3.3. *For any $\delta > 0$, we set:*

$$\tau_\delta(\sigma) := \inf\left\{t \geq 0\ :\ \left|\left|\mathcal{X}_t^N - \varphi_t\left(\overline{x_0}\right)\right|\right|_N > \delta\right\}\,.$$

Then, for any $\delta > 0$ and for any $T \geq 0$, we have the following limit:

$$\lim_{\sigma\to 0}\mathbb{P}\left\{\tau_\delta(\sigma) \leq T\right\} = 0\,.$$

From Proposition 3.3, we immediately deduce the following result about the first entering time in a domain which contains a critical point.

Proposition 3.4. *Let a be a real such that $\nabla V(a) = 0$. Let $\mathcal{D}$ be an open domain which contains $\overline{a} = (a, \cdots, a)$. We assume that the deterministic system $(x_t)_{t\geq 0}$, defined by*

$$x_t = x_0 - \int_0^t \nabla V\left(x_s\right)ds\,,$$

converges toward a. Then, there exists $T > 0$ such that we have the following convergence:

$$\mathbb{P}\left\{E_{\mathcal{D}}(\sigma, N) \leq T\right\} \longrightarrow 1\,,$$

as σ goes to 0.

Proof. We observe the following: $\varphi_t\left(\overline{x_0}\right) = \overline{x_t} = (x_t, \cdots, x_t)$. According to the hypotheses of the proposition, $\varphi_t\left(\overline{x_0}\right)$ converges toward $\overline{a}$ as t goes to infinity. Since $\overline{a} \in \mathcal{D}$, we deduce that there exists $T \geq 0$ such that

$$\varphi_t\left(\overline{x_0}\right) \in \mathcal{D} \quad \text{for any } t \geq T - 1\,.$$

With $\delta := \frac{1}{2}\inf_{z\in\mathcal{D}}||z - \varphi_T\left(\overline{x_0}\right)||$, applying Proposition 3.3 yields the limit

$$\begin{aligned}
\mathbb{P}\left\{E_{\mathcal{D}}(\sigma, N) \leq T\right\} &= 1 - \mathbb{P}\left\{E_{\mathcal{D}}(\sigma, N) > T\right\}\\
&\geq 1 - \mathbb{P}\left\{\mathcal{X}_T^N \notin \mathcal{D}\right\}\\
&\geq 1 - \mathbb{P}\left\{\left|\left|\mathcal{X}_T^N - \varphi_T\left(\overline{x_0}\right)\right|\right|_N > \delta\right\}\\
&\geq 1 - \mathbb{P}\left\{\tau_\delta(\sigma) \leq T\right\}\\
&\longrightarrow 1\,,
\end{aligned}$$

as σ goes to 0. $\square$

We can remark similarly that the time for exiting a domain which does not contain any critical point is finite with large probability.

We now give a definition which is of crucial interest.

Definition 3.5. *Let k be any positive integer. Let $\mathcal{D}$ be an open domain of $\mathbb{R}^k$ and U be a potential of $\mathbb{R}^k$. In the following, we say that $\mathcal{D}$ is stable by the potential U if for any $\xi_0 \in \mathcal{D}$, for any $t \geq 0$, we have $\xi_t \in \mathcal{D}$ with*

$$\xi_t = \xi_0 - \int_0^t \nabla U\left(\xi_s\right) ds\,.$$

We now introduce a new potential which is linked to the potential V.

Definition 3.6. *Let $a \in \mathbb{R}$ be a critical point of V. We introduce the potential W^a defined by*

$$W^a(x) := V(x) + F(x-a)\,.$$

The potential W^a is central in the study of the invariant probabilities of the McKean-Vlasov diffusion, that is the hydrodynamical limit of the mean-field system of particles. Indeed, by Theorem 2.3 in [Tug12a] and by Proposition 1.2 in [Tug13a], we know that δ_a is the small-noise limit of invariant probabilities only if we have the inequality

$$W^a(b) > W^a(a)\,,$$

for any $b \neq a$.

And, see [Tug14], the potential Υ^N admits a large number of local minimizers albeit most of these wells do not correspond to the small-noise limits of invariant probabilities for the McKean-Vlasov diffusion. It has been shown that the diffusion $\mathcal{X}^N$ does not see these wells by looking at the hydrodynamical limit. Here, we show it directly by solving the exit-problem of the system of particles from some domain of attraction.

Definition 3.7. *Let a be a real such that $\overline{a}$ is a local minimizer of Υ^N. By $\mathcal{D}_N(\overline{a})$, we denote the domain of attraction of $\overline{a}$ that is to say the set*

$$\mathcal{D}_N(\overline{a}) := \left\{\mathcal{X} \in \mathbb{R}^N \;\;:\;\; \lim_{t\to\infty} \varphi_t(\mathcal{X}) = \overline{a}\right\}$$

with

$$\varphi_t(\mathcal{X}) = \mathcal{X} - N\int_0^t \nabla\Upsilon^N\left(\varphi_s(\mathcal{X})\right) ds\,.$$

for any $\mathcal{X} \in \mathbb{R}^N$.

More generaly, for any critical point of Υ^N, $\mathcal{X}_0$, we define $\mathcal{D}_N(\mathcal{X}_0)$ in the same way.

We now give a result which means that, under a suitable assumption, a local minimizer of the potential Υ^N does not correspond to the small-noise limit of invariant probabilities of the McKean-Vlasov diffusion.

Proposition 3.8. *Let a be a real such that $\nabla V(a) = 0$. We assume that there exists a real b such that $W^a(b) < W^a(a)$. Let x_0 be a real such that $\overline{x_0} \in \mathcal{D}_n(\overline{a})$, the domain of attraction of the critical point $(a, \cdots, a)$.*
Let $\mathcal{G}$ be any open domain included in $\mathcal{D}_N(\overline{a})$. We assume that it is stable by the potential $N\Upsilon^N$ (see Definition 3.5 for the definition of the stability). We also assume $\overline{\mathcal{G}} \subset \mathcal{D}_N(\overline{a})$.
Then, there exists $H > 0$ such that, for N large enough:

$$\mathbb{P}\left\{\tau_{\mathcal{G}}(\sigma, N) \leq e^{\frac{2H}{\sigma^2}}\right\} \longrightarrow 1\,,$$

as σ goes to 0.

Proof. If N is large enough, a simple computation gives us

$$N\Upsilon^N(b; a; \cdots; a) - N\Upsilon^N(a; \cdots; a) = W^a(b) - W^a(a) + o(1) < 0\,.$$

Consequently, the deterministic system $(\varphi_t(b; a; \cdots; a))_{t\geq 0}$ does not converge to $(a, \cdots, a)$ as t goes to infinity. We deduce that $(b, a, \cdots, a)$ does not belong to $\mathcal{D}_N(\overline{a})$.
We now consider the function from $[0; 1]$ to $\mathbb{R}^N$, $\xi_{a\to b}$, defined by

$$\xi_{a\to b}(t) := (a + t(b - a); a \cdots; a)\ .$$

We observe that: $\xi_{a\to b}(0) = \overline{a} \in \mathcal{G}$. And, $\xi_{a\to b}(1) = (b; a; \cdots; a) \notin \mathcal{D}_N(\overline{a})$ so that $\xi_{a\to b}(1) \notin \mathcal{G}$. Consequently, the path $\xi_{a\to b}$ has at least one intersection with $\partial\mathcal{G}$ in a point of the form

$$(x_N; a; \cdots; a)\ .$$

And, for N large enough, we have

$$N\Upsilon^N(x_N; a; \cdots; a) - N\Upsilon^N(a; \cdots; a) = W^a(x_N) - W^a(a) + o(1)\,.$$

However, $x_N \in [a; b]$. We put:

$$H := \sup_{x\in[a;b]} W^a(x) + 1 - W^a(a)\,.$$

Then, applying Theorem 1.1 yields for any $N \geq 1$:

$$\mathbb{P}\left\{\tau_{\mathcal{G}}(\sigma, N) \leq e^{\frac{2H_N}{\sigma^2}}\right\} \longrightarrow 1$$

as σ goes to 0. Here, $H_N := \inf\limits_{\mathcal{Z}\in\partial\mathcal{G}} N\Upsilon^N(\mathcal{Z}) - N\Upsilon^N(\overline{a})$. Then, for N large enough, we have

$$\begin{aligned} H_N &\leq N\Upsilon^N(x_N; a; \cdots; a) - N\Upsilon^N(a; \cdots; a) \\ &\leq W^a(x_N) - W^a(a) + o(1) < H\,, \end{aligned}$$

which achieves the proof. $\square$

This means that the diffusion is not captiv from the domain of attraction of $\overline{a}$ for large N. Indeed, the time to exit does not depend on the number of particles.

Remark 3.9. *Let us now consider a local minimizer of Υ^N of the form*

$$\mathcal{X}_0 := (a_1; \cdots; a_1; \cdots; a_k; \cdots; a_k)$$

where there are Nr_i elements a_i for any $1 \leq i \leq k$. We put:

$$W^{\mathcal{X}_0}(x) := V(x) + \frac{r_1}{2}\left(A_2(x; a_1) + A_2(a_1; x)\right) + \cdots + \frac{r_k}{2}\left(A_2(x; a_k) + A_2(a_k; x)\right).$$

We assume that there exists $b \in \mathbb{R}$ and $i \in [\![1; k]\!]$ such that

$$W^{\mathcal{X}_0}(b) < W^{\mathcal{X}_0}(a_i).$$

Then, we have a result similar to the previous one. Let $\mathcal{G}$ be any open domain included into $\mathcal{D}_N(\overline{a})$. We assume that it is stable by the potential $N\Upsilon^N$ and that $\overline{\mathcal{G}} \subset \mathcal{D}_N(\overline{a})$.
Then, there exists $H > 0$ such that, for N large enough:

$$\mathbb{P}\left\{\tau_{\mathcal{G}}(\sigma, N) \leq e^{\frac{2H}{\sigma^2}}\right\} \longrightarrow 1$$

as σ goes to 0.

In particular, if there exist $i \neq j$ such that $W^{\mathcal{X}_0}(a_i) \neq W^{\mathcal{X}_0}(a_j)$, the probability which is associated to the point $\mathcal{X}_0$ that is to say $r_1\delta_{a_1} + \cdots + r_k\delta_{a_k}$ does not correspond to the small-noise limit of invariant probabilities for the McKean-Vlasov diffusion. Consequently, we find again the equations (3.12) in [HT10b].

4 Large deviations : Reduction of Hypotheses

The hypotheses for applying the Freidlin-Wentzell theory concern some dynamical behaviour and they are not so easy to prove. In particular, the stability of the domain by the potential generally is tedious to obtain, see [Tug12b]. We here aim to establish sufficient conditions, for obtaining the classical results of large deviations, which deal only with the geometry of the domain $\mathcal{G}$.

Let us briefly justify in which sense we mean that the stability by a domain is not easy to obtain. In [Tug12b], we have solved the exit-problem of the first particle from the domain $\mathcal{D}$. This is equivalent to solve the exit-problem of the whole system of particle from the domain $\mathcal{D} \times \left(\mathbb{R}^d\right)^{N-1}$. However, this domain is not stable by the potential $N\Upsilon^N$. Consequently, we have had to consider the intersection between this domain and the ball of center a and radius κ, κ being small. This new domain is stable by $N\Upsilon^N$. However, to prove so, we need to establish the stability of the ball by $N\Upsilon^N$. Moreover, we also had to prove that the exit-cost from the ball is larger than the exit-cost of the domain $\mathcal{D}\times\left(\mathbb{R}^d\right)^{N-1}$.

However, the computation of the exit-cost from the ball is doable by using the convexity of the potential V and of the potential F. Furthermore, the ball is not stable by $N\Upsilon^N$ without the convexity of the two potentials.

This is why we need to circumvent the difficulty of the stability.

The aim of this section is to give a "new" result of large deviations in $\mathbb{R}^k$, for any $k\in\mathbb{N}^*$.

4.1 Application of Freidlin-Wentzell theory to level sets

We begin by applying the classical results of the Freidlin-Wentzell theory to the level sets because these sets are of crucial interest in the proof of our new result.

Definition 4.1. *Let k be any positive integer and U be a potential on $\mathbb{R}^k$. For any $H\in\mathbb{R}$, we define its H-level set by*

$$\Lambda_H:=\left\{x\in\mathbb{R}^k\;:\;U(x)<H\right\}\,.$$

We can observe that $\Lambda_{H_1}\subset\Lambda_{H_2}$ for any $-\infty\leq H_1\leq H_2\leq+\infty$. Moreover, $\Lambda_{+\infty}=\mathbb{R}^k$.

Furthermore, if the potential U is convex, the set Λ_H is path-connected. But, if U is not convex, Λ_H may not be path-connected. However, we deal with diffusions so the only sets in which we are interested are path-connected.

Definition 4.2. *Let k be any positive integer and U be a potential on $\mathbb{R}^k$. Set $x_0\in\mathbb{R}^k$. From now on, $\mathcal{L}_H(x_0)$ denotes the path-connected subset of the set Λ_H which contains x_0.*

Let us note that $\mathcal{L}_H(x_0)$ is an open set if U is continuous due to the definition of Λ_H.

We now apply the Freidlin-Wentzell theory to these domains.

Proposition 4.3. *Let k be any positive integer. Let U be a potential on $\mathbb{R}^k$. We assume that U is $\mathcal{C}^2$-continuous. We consider the diffusion x^σ defined by*

$$x_t^\sigma=x_0+\sigma\mathcal{B}_t-\int_0^t\nabla U\left(x_s^\sigma\right)ds\,.$$

Let $a\in\mathbb{R}^k$ be a local minimizer of the potential U. We assume that a is the unique critical point in $\overline{\mathcal{L}_H(a)}$ and that $x_0\in\mathcal{L}_H(a)$.

By $\tau_{\mathcal{L}_H(a)}(\sigma,k)$, we denote the first exit-time of the diffusion x^σ from the domain $\mathcal{L}_H(a)$.

Then, for any $\delta>0$, we have

$$\mathbb{P}\left\{e^{\frac{2}{\sigma^2}(H-\delta)}\leq\tau_{\mathcal{L}_H(a)}(\sigma,k)\leq e^{\frac{2}{\sigma^2}(H+\delta)}\right\}\longrightarrow 1$$

as σ goes to 0.

Proof. The domain $\mathcal{L}_H(a)$ is open since U is continuous. Moreover, for any $x_0 \in \mathcal{L}_H(a)$, $x_t \in \mathcal{L}_H(a)$ where the deterministic system $(x_t)_{t\geq 0}$ is defined by

$$x_t = x_0 - \int_0^t \nabla U\left(x_s\right) ds\,.$$

Indeed, we observe:

$$\begin{aligned}\frac{d}{dt}U(x_t) &= \left\langle \frac{d}{dt}x_t\,;\,\nabla U(x_t)\right\rangle \\ &= -\left|\left|\nabla U(x_t)\right|\right|^2 \leq 0\,.\end{aligned}$$

This proves that the domain $\mathcal{L}_H(a)$ is stable by U. Since a is the unique critical point of U on $\overline{\mathcal{L}_H(a)}$, we deduce the convergence of x_t toward a as t goes to infinity, for any $x_0 \in \overline{\mathcal{L}_H(a)}$.
Finally, by construction, for any $z \in \partial\mathcal{L}_H(a)$, we have $U(z) = H$ so that

$$\inf_{z\in\partial\mathcal{L}_H(a)} U(z) = H < \infty\,.$$

Applying Theorem 1.1 achieves the proof. □

4.2 New theorem

We first give the theorem. The next three subsections aim to prove it.

Theorem 4.4. *Let k be any positive integer. We consider a potential U on $\mathbb{R}^k$. We assume that U is $\mathcal{C}^2$-continuous. We consider the diffusion x^σ defined by*

$$x_t^\sigma = x_0 + \sigma\mathcal{B}_t - \int_0^t \nabla U\left(x_s^\sigma\right) ds\,.$$

Let $a \in \mathbb{R}^k$ be a local minimizer of the potential U. Without any loss of generality, we assume $U(a) = 0$. We consider an open domain $\mathcal{G}$ which satisfies the following assumptions:

- *The point a is in $\mathcal{G}$.*
- *The quantity $H := \inf\limits_{z\in\partial\mathcal{G}} U(z)$ is finite.*
- *The set $\partial\mathcal{G} \cap \partial\mathcal{L}_H(a)$ is not empty.*
- *There exists $\kappa_0 > 0$ such that the potential U admits a unique critical point in $\mathcal{L}_{H+\kappa_0}(a)$.*

By $\tau_\mathcal{G}(\sigma, k)$, we denote the first exit-time of the diffusion x^σ from the domain $\mathcal{G}$. Then, the three following results hold.

i) *For any $\delta > 0$, we have*

$$\lim_{\sigma\to 0} \mathbb{P}\left\{e^{\frac{2}{\sigma^2}(H-\delta)} \leq \tau_\mathcal{G}(\sigma, k)\right\} = 1\,.$$

ii) *For any $\delta > 0$, we have*

$$\lim_{\sigma \to 0} \mathbb{P}\left\{\tau_{\mathcal{G}}(\sigma, k) \leq e^{\frac{2}{\sigma^2}(H+\delta)}\right\} = 1\,.$$

iii) *If $\mathcal{N} \subset \partial\mathcal{G}$ is such that $\inf_{z \in \mathcal{N}} U(z) > \inf_{z \in \partial\mathcal{G}} U(z)$ then, we have:*

$$\lim_{\sigma \to 0} \mathbb{P}\left\{x^{\sigma}_{\tau_{\mathcal{G}}(\sigma,k)} \in \mathcal{N}\right\} = 0\,.$$

In other words, we find the same results than the one of the Freidlin-Wentzell theory. We could also obtain easily:

$$\lim_{\sigma \to 0} \frac{\sigma^2}{2} \log\left\{\mathbb{E}\left[\tau_{\mathcal{G}}(\sigma, k)\right]\right\} = H\,.$$

4.3 Proof of the lower-bound

We take the assumptions of Theorem 4.4 and we prove i) that is to say the limit, for any $\delta > 0$:

$$\mathbb{P}\left\{e^{\frac{2}{\sigma^2}(H-\delta)} \leq \tau_{\mathcal{G}}(\sigma, k)\right\} \longrightarrow 1$$

as σ goes to 0. For doing so, we prove, for any $\delta > 0$, the limit:

$$\mathbb{P}\left\{\tau_{\mathcal{G}}(\sigma, k) < e^{\frac{2}{\sigma^2}(H-\delta)}\right\} \longrightarrow 0$$

as σ goes to 0. We observe that we have

$$\mathcal{L}_{H-\frac{\delta}{2}}(a) \subset \mathcal{G}\,,$$

for any $\delta > 0$. Consequently, we have:

$$\tau_{\mathcal{G}}(\sigma, k) \geq \tau_{\mathcal{L}_{H-\frac{\delta}{2}}(a)}(\sigma, k)\,,$$

so that

$$\mathbb{P}\left\{\tau_{\mathcal{G}}(\sigma, k) < e^{\frac{2}{\sigma^2}(H-\delta)}\right\} \leq \mathbb{P}\left\{\tau_{\mathcal{L}_{H-\frac{\delta}{2}}(a)}(\sigma, k) < e^{\frac{2}{\sigma^2}(H-\delta)}\right\}\,.$$

However, according to Proposition 4.3, for any $\gamma > 0$, we have:

$$\mathbb{P}\left\{e^{\frac{2}{\sigma^2}\left(H-\frac{\delta}{2}-\gamma\right)} \leq \tau_{\mathcal{L}_{H-\frac{\delta}{2}}(a)}(\sigma, k)\right\} \longrightarrow 1$$

as σ goes to 0. Taking $\gamma := \frac{\delta}{2}$ achieves the proof.

4.4 Proof of the upper-bound

We take the assumptions of Theorem 4.4 and we prove ii) that is to say the limit, for any $\delta > 0$:

$$\mathbb{P}\left\{\tau_{\mathcal{G}}(\sigma, k) \leq e^{\frac{2}{\sigma^2}(H+\delta)}\right\} \longrightarrow 1$$

as σ goes to 0. For doing so, we prove, for any $\delta > 0$, the limit:

$$\mathbb{P}\left\{\tau_{\mathcal{G}}(\sigma, k) > e^{\frac{2}{\sigma^2}(H+\delta)}\right\} \longrightarrow 0$$

as σ goes to 0.

Here, it is not as simple as in the previous paragraph. Indeed, we do not have immediately $\mathcal{G} \subset \mathcal{L}_{H+\frac{\delta}{2}}(a)$ if $\mathcal{G}$ is not the level set $\mathcal{L}_H(a)$. If it is, we simply apply Proposition 4.3. From now on, we assume that $\mathcal{G}$ is not the domain $\mathcal{L}_H(a)$.

Consequently, we slightly modify the domain $\mathcal{G}$ so that the diffusion leaves almost surely $\mathcal{G}$ before exiting from this new domain.

To do so, we need to give some definitions.

Definition 4.5. *We introduce the descending deterministic system:*

$$\psi_t(x) = x - \int_0^t \nabla U\left(\psi_s(x)\right) ds\,.$$

And, we introduce the ascending deterministic system:

$$\widehat{\psi_t}(x) = x + \int_0^t \nabla U\left(\widehat{\psi_s}(x)\right) ds\,.$$

We observe that we have $\psi_t\left(\widehat{\psi_t}(x)\right) = x$, for any $x \in \mathbb{R}^k$ and $t \geq 0$.

Definition 4.6. *For any x in $\partial\mathcal{L}_{H+\frac{\delta}{2}}$, we consider $T_\delta(x) > 0$ such that*

$$\widehat{\psi_{T_\delta(x)}}(x) \in \partial\mathcal{L}_{H+\delta}(a)\,.$$

We now introduce the distance between the domains $\mathcal{G}$ and $\partial\mathcal{L}_{H+\frac{\delta}{2}}(a)$.

Definition 4.7. *For any $\delta > 0$, we put:*

$$\rho(\delta) := \frac{1}{2} \sup_{\substack{x \in \partial\mathcal{L}_{H+\frac{\delta}{2}}(a) \\ x \notin \mathcal{G}}} \inf_{y \in \partial\mathcal{G}} d\left(x; y\right)\,.$$

Let us note that nothing ensures us, *a priori*, that $\rho(\delta)$ is positive.

Remark 4.8. *However, $\mathcal{G}^c \cap \partial\mathcal{L}_{H+\frac{\delta}{2}}(a) \neq \emptyset$ if $\delta > 0$ is small enough. Indeed, the two sets $\partial\mathcal{G}$ and $\partial\mathcal{L}_H(a)$ have a non empty intersection and U has a unique critical point on the domain $\partial\mathcal{L}_{H+\kappa_0}(a)$ for some positive κ_0. And, if the set $\mathcal{G}^c \cap \partial\mathcal{L}_{H+\frac{\delta}{2}}(a)$ was empty, it would imply that each point of $\partial\mathcal{G} \cap \partial\mathcal{L}_H(a)$ is a local maximizer so a critical point.*

The new domain that we consider is an enlargement of $\mathcal{L}_{H+\frac{\delta}{2}}(a)$:

$$\mathcal{D}_\delta := \mathcal{L}_{H+\frac{\delta}{2}}(a) \bigcup \left[\bigcup_{\substack{z\in\partial\mathcal{L}_{H+\frac{\delta}{2}}(a) \\ z\in\mathcal{G}+\mathbb{B}(\rho(\delta))}} \left\{ \widehat{\psi_t}(z)\,;\, 0 \leq t < T_\delta(z) \right\} \right]$$

where $\mathbb{B}(\rho(\delta))$ is the ball of center 0 and radius $\rho(\delta)$. We now apply Theorem 1.1 to this new domain.
To do so, we need to check that the assumptions of the theorem are satisfied.

Lemma 4.9. *There exists $\delta_1 > 0$ such that for any $0 < \delta < \delta_1$, the potential U admits a unique critical point in the domain $\mathcal{D}_\delta$.*

Proof. By definition of the domain, we have the following inclusion:

$$\mathcal{D}_\delta \subset \mathcal{L}_{H+\delta}(a)\,.$$

Due to the fourth hypothesis of Theorem 4.4, if $\delta > 0$ is small enough, the potential U admits a unique critical point in the domain $\mathcal{L}_{H+\delta}(a)$ so in the domain $\mathcal{D}_\delta$. □

We now prove the stability of the domain and the convergence of the deterministic system starting from any boundary point of $\mathcal{D}_\delta$.

Lemma 4.10. *There exists $\delta_2 > 0$ such that for any $0 < \delta < \delta_2$, we have:*

- *For any $y_0 \in \mathcal{D}_\delta$, for any $t > 0$, $\psi_t(y_0) \in \mathcal{D}_\delta$ that is to say $\mathcal{D}_\delta$ is stable by U.*
- *For any $y_0 \in \overline{\mathcal{D}_\delta}$, $\psi_t(y_0)$ converges toward a as t goes to infinity.*

Proof. By construction, the domain $\mathcal{D}_\delta$ is stable by U.
Now, let y_0 be in $\overline{\mathcal{D}_\delta}$. If δ is small enough, the potential U admits a unique critical point in $\mathcal{L}_{H+\delta}(a)$ so that the deterministic system $\psi_t(y_0)$ converges to a as t goes to infinity. □

Lemma 4.11. *There exists $\delta_3 > 0$ such that for any $0 < \delta < \delta_3$, we have:*

$$\inf_{z\in\partial\mathcal{D}_\delta} U(z) = H + \frac{\delta}{2}\,.$$

Proof. Let x be in $\mathcal{D}_\delta$ such that $x \notin \mathcal{L}_{H+\frac{\delta}{2}}(a)$. Then, we remark that: $U(x) \geq H + \frac{\delta}{2}$.
Moreover, if δ is small enough, we observe that $\partial\mathcal{L}_{H+\frac{\delta}{2}}(a) \cap (\mathcal{G} + \mathbb{B}(\rho(\delta))) \neq \partial\mathcal{L}_{H+\frac{\delta}{2}}(a)$. Consequently, the following holds:

$$\partial\mathcal{D}_\delta \cap \partial\mathcal{L}_{H+\frac{\delta}{2}}(a) \neq \emptyset\,.$$

And, for any $x \in \partial\mathcal{L}_{H+\frac{\delta}{2}}(a)$, we have $U(x) = H + \frac{\delta}{2}$. We deduce that the infimum of the potential U on the boundary of $\mathcal{D}_\delta$ is $H + \frac{\delta}{2}$. □

We now are able to apply Theorem 1.1 if we prove that the domain $\mathcal{D}_\delta$ is open for sufficiently small δ. It is the tedious part of the proof.

Lemma 4.12. *There exists $\delta_4 > 0$ such that for any $0 < \delta < \delta_4$, the domain $\mathcal{D}_\delta$ is open.*

Proof. Due to the continuity of the potential U, we know that $\mathcal{L}_{H+\frac{\delta}{2}}(a)$ is open. Consequently, it is sufficient to prove that the points in

$$\bigcup_{\substack{z\in\partial\mathcal{L}_{H+\frac{\delta}{2}}(a)\\ z\in\mathcal{G}+\mathbb{B}(\rho(\delta))}} \left\{\widehat{\psi_t}(z)\,;\,0\le t<T_\delta(z)\right\}$$

are in the interior of $\mathcal{D}_\delta$.
Let z be in $\partial\mathcal{L}_{H+\frac{\delta}{2}}(a)$ such that $d(z;\mathcal{G}) < \rho(\delta)$. Let t be a positive real which is less than $T_\delta(z)$. By definition, $\widehat{\psi_t}(z) \in \mathcal{L}_{H+\delta}(a)$. Since $\mathcal{L}_{H+\delta}(a)$ is open, there exists $\kappa_1 > 0$ such that the ball of center $\widehat{\psi_t}(z)$ and radius κ_1, $\mathbb{B}_{\widehat{\psi_t}(z)}(\kappa_1)$, is included into $\mathcal{L}_{H+\delta}(a)$.
We now proceed a *reducto ad absurdum.* We assume that for any $\kappa < \kappa_1$, there exists $x_\kappa \in \mathbb{B}_{\widehat{\psi_t}(z)}(\kappa)$ such that $x_\kappa \notin \mathcal{D}_\delta$. As the domain $\mathcal{L}_{H+\delta}(a)$ is stable by U, $\psi_s(x_\kappa) \in \mathcal{L}_{H+\delta}(a)$ for any $s \ge 0$.
Moreover, by taking δ sufficiently small, we know by Lemma 4.9 that the potential U has a unique critical point on the domain $\mathcal{L}_{H+\delta}(a)$. Consequently, there exists $T_\kappa > 0$ such that $U\left(\psi_{T_\kappa}(x_\kappa)\right) = H + \frac{\delta}{2}$. Thus, since $x_\kappa \notin \mathcal{D}_\delta$, we obtain:

$$d\left(\psi_{T_\kappa}(x_\kappa);\mathcal{G}\right) \ge \rho(\delta)\,. \tag{3}$$

However, since U is continuous, the limit

$$\lim_{\kappa\to 0}\ \sup_{z\in\mathbb{B}_{\widehat{\psi_{T_\kappa}}(z)}(\kappa)} d\left(\psi_{T_\kappa}(x_\kappa);\mathcal{G}\right) = 0$$

yields that Inequality (3) is absurd for sufficiently small κ. So, we deduce the existence of $\kappa > 0$ such that

$$\mathbb{B}_{\widehat{\psi_t}(z)}(\kappa) \subset \mathcal{D}_\delta\,.$$

This proves that the points in

$$\bigcup_{\substack{z\in\partial\mathcal{L}_{H+\frac{\delta}{2}}(a)\\ z\in\mathcal{G}+\mathbb{B}(\rho(\delta))}} \left\{\widehat{\psi_t}(z)\,;\,0\le t<T_\delta(z)\right\}$$

are in the interior of $\mathcal{D}_\delta$ if δ is small enough. Consequently, for sufficiently small δ, the domain $\mathcal{D}_\delta$ is open. □

Let us now turn on the proof of ii). We put $\delta_0 := \min\{\delta_1;\delta_2;\delta_3;\delta_4\}$. We take from now on $\delta < \delta_0$ so that the results of Lemma 4.9, Lemma 4.10, Lemma

4.11 and Lemma 4.12 hold.
Consequently, we can apply Theorem 1.1. Particularly, for any $\gamma > 0$, we have the limit:

$$\mathbb{P}\left\{\tau_{\mathcal{D}_\delta}(\sigma,k) > e^{\frac{2}{\sigma^2}\left(H+\frac{\delta}{2}+\gamma\right)}\right\} \longrightarrow 0 \tag{4}$$

as σ goes to 0, $\tau_{\mathcal{D}_\delta}(\sigma,k)$ being the exit-time of the diffusion x^σ from the domain $\mathcal{D}_\delta$.
Let us now introduce the domain: $\mathcal{N}_\delta := \partial\mathcal{D}_\delta \cap \mathcal{G}$. By construction of $\mathcal{D}_\delta$, we have $U(x) > H + \frac{\delta}{2}$ for any $x \in \mathcal{N}_\delta$. Moreover, we have the inequality:

$$\inf_{z\in\mathcal{N}_\delta} U(z) > H + \frac{\delta}{2}\,.$$

Consequently, according to the exit-location result of Theorem 1.1, we have the limit:

$$\mathbb{P}\left\{x^\sigma_{\tau_{\mathcal{D}_\delta}(\sigma,k)} \in \mathcal{N}_\delta\right\} \longrightarrow 0 \tag{5}$$

as σ goes to 0. Thus, we deduce:

$$\begin{aligned}
&\mathbb{P}\left\{\tau_{\mathcal{G}}(\sigma,k) > e^{\frac{2}{\sigma^2}(H+\delta)}\right\}\\
\leq&\mathbb{P}\left\{\tau_{\mathcal{D}_\delta}(\sigma,k) > e^{\frac{2}{\sigma^2}(H+\delta)}\right\} + \mathbb{P}\left\{\tau_{\mathcal{G}}(\sigma,k) > e^{\frac{2}{\sigma^2}(H+\delta)} \geq \tau_{\mathcal{D}_\delta}(\sigma,k)\right\}\\
\leq&\mathbb{P}\left\{\tau_{\mathcal{D}_\delta}(\sigma,k) > e^{\frac{2}{\sigma^2}(H+\delta)}\right\} + \mathbb{P}\left\{x^\sigma_{\tau_{\mathcal{D}_\delta}(\sigma,k)} \in \mathcal{G}\right\}\\
\leq&\mathbb{P}\left\{\tau_{\mathcal{D}_\delta}(\sigma,k) > e^{\frac{2}{\sigma^2}\left(H+\frac{\delta}{2}+\frac{\delta}{2}\right)}\right\} + \mathbb{P}\left\{x^\sigma_{\tau_{\mathcal{D}_\delta}(\sigma,k)} \in \mathcal{N}_\delta\right\}\,.
\end{aligned}$$

By taking $\gamma := \frac{\delta}{2}$ in Limit (4), we deduce that the first term goes to 0 as σ goes to 0. The second term converges to 0 as σ goes to 0 according to Limit (5). This achieves the proof.

4.5 Proof of the exit-location

We take the assumptions of Theorem 4.4 and we prove iii). Let $\mathcal{N} \subset \partial\mathcal{G}$ such that $\inf_{z\in\mathcal{N}} U(z) > H$. Let us prove:

$$\mathbb{P}\left\{x^\sigma_{\tau_{\mathcal{G}}(\sigma,k)} \in \mathcal{N}\right\} \longrightarrow 0$$

as σ goes to 0.
Let κ be a positive real such that $\inf_{z\in\mathcal{N}} U(z) = H + 3\kappa$. We observe the following:

$$\mathcal{N} \cap \mathcal{L}_{H+2\kappa}(a) = \emptyset\,.$$

Consequently, we have:

$$\begin{aligned}\mathbb{P}\left\{x^\sigma_{\tau_{\mathcal{G}}(\sigma,k)}\in\mathcal{N}\right\}\leq&\mathbb{P}\left\{x^\sigma_{\tau_{\mathcal{G}}(\sigma,k)}\notin\mathcal{L}_{H+2\kappa}(a)\right\}\\ \leq&\mathbb{P}\left\{\tau_{\mathcal{G}}(\sigma,k)>\tau_{\mathcal{L}_{H+2\kappa}(a)}(\sigma,k)\right\}\\ \leq&\mathbb{P}\left\{\tau_{\mathcal{G}}(\sigma,k)>e^{\frac{2}{\sigma^2}(H+\kappa)}\right\}\\ &+\mathbb{P}\left\{\tau_{\mathcal{L}_{H+2\kappa}(a)}(\sigma,k)<e^{\frac{2}{\sigma^2}(H+\kappa)}\right\}\,.\end{aligned}$$

The first term goes to 0 as σ goes to 0 due to the result ii) in Theorem 4.4. The second term goes to 0 as σ goes to 0 thanks to Proposition 4.3. We deduce that $\mathbb{P}\left\{x^\sigma_{\tau_{\mathcal{G}}(\sigma,k)}\in\mathcal{N}\right\}$ converges to 0 as σ goes to 0.

5 Main results

We now present the main results of the paper. We are interested in the first exit-time (and the corresponding exit-location) of one particle among the whole system of particles (I). We also solve the exit-problem of a tagged particle, the first one, without any loss of generality.

We now define the exit-cost of the diffusion $\mathcal{X}^N$ from a domain $\mathcal{G}\subset\mathbb{R}^N$ which contains a local minimizer and no other critical point of the potential Υ^N. Let us denote this local minimizer by a from now on. Without any loss of generality, we assume $V(a)=0$ so that $\Upsilon^N(a;\cdots;a)=0$. The aim of this assumption is to simplify the writting and the computations.

Definition 5.1. *Let $\mathcal{G}$ be an open domain of $\mathbb{R}^N$. We define its exit-cost by:*

$$H^N(\mathcal{G})=\inf_{\mathcal{Z}\in\partial\mathcal{G}}N\Upsilon(\mathcal{Z})\,.$$

Let us observe that if we did not assume $V(a)\neq 0$, we would have another definition of the exit-cost:

$$H^N(\mathcal{G}):=\inf_{\mathcal{Z}\in\partial\mathcal{G}}N\Upsilon^N(\mathcal{Z})-N\Upsilon^N(a;\cdots;a)\,.$$

The assumption that we take on the domain $\mathcal{D}$ is the following.

Hypothesis 5.2. *The potential $V+\frac{\alpha}{2}(x-a)^2$ is uniformly strictly convex on $\mathcal{D}$.*

Here, we are interested in the exit-problem of any particle. It is equivalent to study the exit-problem of the diffusion $\mathcal{X}^N$ from the domain $\mathcal{G}_N:=\mathcal{D}^N$. Let us first compute the exit-cost $H_N:=H^N(\mathcal{G}_N)$ from this domain.

Lemma 5.3. *We have the following limit:*

$$\lim_{N\to\infty}H_N=H:=\inf_{z\in\partial\mathcal{D}}\left[V(z)+F(z-a)\right]=\inf_{z\in\partial\mathcal{D}}W^a(z)\,.$$

Moreover, if $\mathcal{N} \subset \partial\mathcal{D}$ is such that $\inf_{z\in\mathcal{N}} W^a(z) > \inf_{z\in\partial\mathcal{D}} W^a(z)$, we have the inequality

$$\inf_{\mathcal{N}\times\mathcal{D}^{N-1}} N\Upsilon^N > H_N\,,$$

if N is large enough.

Proof. **Step 1.** Due to the exchangeability of the particles, to compute H_N is equivalent to compute

$$\inf_{\partial\mathcal{D}\times\mathcal{D}^{N-1}} N\Upsilon^N\,.$$

For any $z \in \partial\mathcal{D}$, we introduce the potential ξ_N^z on $\mathbb{R}^{N-1}$ defined by

$$\xi_N^z\left(X_2;\cdots;X_N\right) := N\Upsilon^N\left(z;X_2;\cdots;X_N\right)\,.$$

We aim to compute

$$\inf_{z\in\partial\mathcal{D}}\inf_{(X_2,\cdots,X_N)\in\mathcal{D}^{N-1}} \xi_N^z\left(X_2;\cdots;X_N\right)\,.$$

First, we look at the critical points and then, we use the recent results in [HT14] to show that the minimum is reached in these critical points.

Step 2. For any $1 \le i \le N$, we have:

$$\frac{\partial}{\partial X_i}\xi_N^z\left(X_2;\cdots;X_N\right) = \nabla V\left(X_i\right) + \frac{1}{N}\sum_{j=1}^N \nabla F\left(X_i - X_j\right)\,.$$

Due to Assumption 5.2, if $\left(X_2^0,\cdots,X_N^0\right)$ is a critical point of ξ_N^z, we have: $X_i^0 = X_k^0 = x_N(z)$. Furthermore, $x_N(z)$ satisfies the following equation:

$$\nabla V\left(x_N(z)\right) + \frac{1}{N}\nabla F\left(x_N(z) - z\right) = 0\,.$$

According to the implicit functions theorem, this equation admits a solution on $\mathcal{D}$ since $\nabla^2 V(a) > 0$. Moreover, due to Assumption 5.2, the potential $x \mapsto V(x) + \frac{1}{N}\left[F(x-z) + (N-1)F(x - x_N(z))\right]$ has a unique critical point on $\mathcal{D}$ which implies the uniqueness of the solution for N large enough.
Furthermore, a limited expansion gives us

$$x_N(z) = a + \frac{1}{N}\left(\nabla^2 V(a)\right)^{-1}\nabla F(z-a) + \frac{f_N(z)}{N}\,,$$

f_N being a bounded function from $\partial\mathcal{D}$ to $\mathbb{R}$ such that $f_N(z) \longrightarrow 0$ as N goes to infinity. Thus, we obtain:

$$\xi_N^z\left(x_N(z);\cdots;x_N(z)\right) = V(z) + F(z-a) + g_N(z)\,,$$

where g_N is a function from $\partial\mathcal{D}$ to $\mathbb{R}$ such that

$$\sup_{z\in\partial\mathcal{D}}|g_N(z)| \longrightarrow 0$$

as N goes to infinity.

Step 3. Let us assume in this step that, for any $z \in \partial\mathcal{D}$, the minimum of the potential ξ_N^z is not reached on the boundary. We immediately deduce:

$$\begin{aligned}
&\inf_{\mathcal{N}\times\mathcal{D}^{N-1}} N\Upsilon^N \\
&= \inf_{z_1\in\mathcal{N}} \inf_{(X_2,\cdots,X_N)\in\mathcal{D}^{N-1}} \xi_N^{z_1}\left(X_2;\cdots;X_N\right) \\
&= \inf_{z_1\in\mathcal{N}} \left(W^a(z_1)+g_N(z_1)\right) \\
&> \inf_{z_2\in\partial\mathcal{D}} \left(W^a(z_2)+g_N(z_2)\right) ,
\end{aligned}$$

if N is large enough and if $\mathcal{N}$ is such that $\inf\limits_{\mathcal{N}} W^a > \inf\limits_{\partial\mathcal{D}} W^a$. Also, we have:

$$\begin{aligned}
&\inf_{\partial\mathcal{D}\times\mathcal{D}^{N-1}} N\Upsilon^N \\
&= \inf_{z_2\in\mathcal{N}} \inf_{(X_2,\cdots,X_N)\in\mathcal{D}^{N-1}} \xi_N^{z_2}\left(X_2;\cdots;X_N\right) \\
&= \inf_{z_2\in\mathcal{N}} \left(W^a(z_2)+g_N(z_2)\right) \\
&\longrightarrow \inf_{\partial\mathcal{D}} W^a = H ,
\end{aligned}$$

as N goes to infinity.

Step 4. It is now sufficient to prove that the minimum on the boundary of the potential ξ_N^z is larger than (or equal to) the minimum on the whole set to achieve the proof. To do so, we remark that $\mathcal{D}^N$ is included into $\mathcal{D}\times\left(\mathbb{R}^d\right)^{N-1}$. Let us show that the minimum of the potential $N\Upsilon^N$ on $\partial\left(\mathcal{D}\times\left(\mathbb{R}^d\right)^{N-1}\right)$ is larger than (or equal to) H_N.

Thanks to the main result in [HT14], we know that the stochastic process $\left(X_t^{1,N}\right)_t$ satisfies a large deviation principle with a good rate function J_N which converges, as N goes to infinity, toward

$$J_\infty(f) := \frac{1}{4}\int_0^T \|\dot{f}(t)+\nabla V\left(f(t)\right)+\nabla F\left(f(t)-\Psi_\infty^x(t)\right)\|^2\,dt, \tag{6}$$

if $f \in \mathcal{H}_x$ and otherwise $J_\infty(f) := +\infty$. Here the function Ψ_∞^x satisfies the following ordinary differential equation:

$$\Psi_\infty^x(t) = x - \int_0^t \nabla V\left(\Psi_\infty^x(s)\right)ds, \qquad x\in\mathbb{R}^d\,. \tag{7}$$

Classical computation in large deviation achieves the proof. Let us remark that the same holds if one consider $\mathcal{N}\subset\partial\mathcal{D}$

□

To apply Theorem 4.4, we still need to check an hypothesis: the uniqueness of the critical point of the potential $N\Upsilon^N$ on the domain $\mathcal{L}_{H_N+\kappa}\left(\overline{a}\right)$ for κ sufficiently small. This is the aim of the following lemma.

Lemma 5.4. *For any $N \in \mathbb{N}$, there exists a compact $\mathcal{K}_N$ of $\mathbb{R}^N$ which contains $\overline{\mathcal{D}}^N$ in its interior such that the potential $N\Upsilon^N$ has a unique critical point on $\mathcal{K}_N$ that is $\overline{a} = (a, \cdots, a)$. Moreover, if N is sufficiently large, there exists $\kappa_0 > 0$ such that $\mathcal{L}_{H_N+\kappa}(\overline{a}) \subset \mathcal{K}_N$, where $\mathcal{L}_{H_N+\kappa}(\overline{a})$ is the level set associated to the potential $N\Upsilon^N$.*

Proof. The potential $V + \frac{\alpha}{2}(x-a)^2$ is uniformly strictly convex on the domain $\mathcal{D}$. So, there exists a compact $\mathcal{K} \subset \mathbb{R}$ such that $\mathcal{K}$ contains $\overline{\mathcal{D}}$ in its interior and such that it is uniformly strictly convex on the compact $\mathcal{K}$.
Let $\left(X_1^0, \cdots, X_N^0\right)$ be a critical point of the potential $N\Upsilon^N$ on the domain $\mathcal{K}_N := \mathcal{K}^N$. Then, we have:

$$\nabla V\left(X_i^0\right) + \frac{1}{N}\sum_{j=1}^N \nabla F\left(X_i^0 - X_j^0\right) = 0 \tag{8}$$

for any $1 \leq i \leq N$. The convexity of the potential $V + \frac{1}{N}\sum_{j=1}^N F\left(. - X_j^0\right)$ on the compact $\mathcal{K}$ implies $X_i^0 = X_j^0 = x_N$. Thus, Equation (8) gives us

$$\nabla V(x_N) = 0\,,$$

which implies $x_N = a$. We deduce $\left(X_1^0, \cdots, X_N^0\right) = (a, \cdots, a)$. Consequently, the potential $N\Upsilon^N$ has a unique critical point on the compact $\mathcal{K}_N$.
We put $\widehat{H_N} := H^N(\mathcal{K}_N)$. By proceeding like in the proof of Lemma 5.3, we obtain the following convergence as N goes to infinity:

$$\widehat{H_N} \longrightarrow \widehat{H} := \inf_{z \in \partial\mathcal{K}} \left(V(z) + F(z-a)\right) > H\,.$$

By putting $\kappa := \frac{1}{2}\left(\widehat{H} - H\right)$, we deduce $\mathcal{L}_{H_N+\kappa}(\overline{a}) \subset \mathcal{K}_N$ if N is large enough. □

Let us note that Lemma 5.4 is strongly based on the local convexity of V. However, we can obtain better result without this assumption.

Remark 5.5. *If α is large enough, the potential $V + \frac{1}{N}\sum_{j=1}^N F\left(. - X_j^0\right)$ is convex on any domain and we can thus apply Lemma 5.4.*

Conversely, if $F(x) := \frac{\alpha}{2}x^2$, then having $\alpha < \alpha_c < 0$, the potential $-V - \frac{1}{N}\sum_{j=1}^N F\left(. - X_j^0\right)$ is convex so there is a unique point x_0 such that $-\nabla V(x_0) - \frac{1}{N}\sum_{j=1}^N \nabla F\left(x_0 - X_j^0\right) = 0$. Immediately, we obtain the statement of Lemma 5.4.

Now, we can apply Theorem 4.4 and obtain the result.

Theorem 5.6. *Let $a \in \mathbb{R}$ be a local minimizer of the potential V. Let $\mathcal{D}$ be an open domain which contains a and which satisfies Hypothesis 5.2.*
By $\tau_{\mathcal{D}^N}(\sigma, N)$, we denote the first exit-time of the diffusion $\mathcal{X}^N$ from the domain $\mathcal{D}^N$. If N is large enough, we have:

i) *For any $\delta > 0$, we have the following limit as σ goes to 0 :*

$$\mathbb{P}\left\{e^{\frac{2}{\sigma^2}(H_N-\delta)} \leq \tau_{\mathcal{D}^N}(\sigma, N) \leq e^{\frac{2}{\sigma^2}(H_N+\delta)}\right\} \longrightarrow 1\,,$$

with $\lim_{N\to\infty} H_N = H := \inf_{z\in\partial\mathcal{D}} W^a(z)$.

ii) *If $\mathcal{N} \subset \partial\mathcal{D}$ is such that $\inf_{\mathcal{N}} W^a > H$, we have the following result on the exit-location:*

$$\lim_{\sigma\to 0} \mathbb{P}\left\{X^{1,N,\sigma}_{\tau_{\mathcal{D}^N}(\sigma,N)} \in \mathcal{N}\right\} = 0\,.$$

Proof. **Step 1.** The domain $\mathcal{D} \times \left(\mathbb{R}^d\right)^{N-1} \subset \mathbb{R}^N$ is open.
Step 2. According to Lemma 5.3, we have the convergence of H_N to H as N goes to infinity. Consequently, for sufficiently large N, $H_N < \infty$.
Step 3. According to Lemma 5.4, the potential $N\Upsilon^N$ has a unique critical point on $\mathcal{L}_{H_N+\kappa}(\overline{a})$ for sufficiently small $\kappa > 0$ and sufficiently large N.
Step 4. Let us verify $\partial\mathcal{D}^N \cap \mathcal{L}_{H_N}(\overline{a}) \neq \emptyset$. According to the proof of Lemma 5.3,

$$(z, x_N(z), \cdots, x_N(z)) \in \partial\mathcal{D}^N \quad \text{and} \quad (z, x_N(z), \cdots, x_N(z)) \in \partial\mathcal{L}_{H_N}\,.$$

Step 5. We apply Theorem 4.4. Then, by Lemma 5.3, we have the inequality

$$\inf_{\mathcal{N}\times(\mathbb{R}^d)^{N-1}} N\Upsilon^N > H_N\,,$$

if N is large enough and if $\mathcal{N}$ is such that $\inf_{\mathcal{N}} W^a > H$. □

We now give a similar result concerning the exit-time of the first particle. We do not provide the proof because it is similar. We can not apply the result to $\mathcal{D} \times \left(\mathbb{R}^d\right)^{N-1}$ so we apply it to a domain of the form $\mathcal{D} \times (\mathcal{D}')^{N-1}$ where $\mathcal{D}'$ is an open domain containing $\overline{\mathcal{D}}$ and satisfying the same hypotheses.

Theorem 5.7. *Let $a \in \mathbb{R}$ be a local minimizer of the potential V. Let $\mathcal{D}$ be an open domain which contains a and which satisfies Hypothesis 5.2.*
By $\tau_{\mathcal{D}^N}(\sigma, N)$, we denote the first exit-time of the diffusion $X^{1,N}$ from the domain $\mathcal{D}$. If N is large enough, we have:

i) *For any $\delta > 0$, we have the following limit as σ goes to 0 :*

$$\mathbb{P}\left\{e^{\frac{2}{\sigma^2}(\widehat{H_N}-\delta)} \leq \tau_{\mathcal{D}^N}(\sigma, N) \leq e^{\frac{2}{\sigma^2}(\widehat{H_N}+\delta)}\right\} \longrightarrow 1\,,$$

with $\lim_{N\to\infty} \widehat{H_N} = H := \inf_{z\in\partial\mathcal{D}} W^a(z)$.

ii) *If $\mathcal{N} \subset \partial\mathcal{D}$ is such that $\inf_{\mathcal{N}} W^a > H$, we have the following result on the exit-location:*

$$\lim_{\sigma\to 0} \mathbb{P}\left\{X^{1,N,\sigma}_{\tau_{\mathcal{D}^N}(\sigma,N)} \in \mathcal{N}\right\} = 0\,.$$

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